ON THE PROPAGATION OF ELASTIC-PLASTIC WAVES OWING TO COMBINED LOADING

(O RASPROSTRANENII UPRUGO-PLASTICHESKIKH VOLN PRI SLOZHNOM NAGRUZHENII)

PMM Vol.22, No.6, 1958, pp. 759-765

Kh.A. RAKHMATULIN
(Moscow)

(Received 20 March 1958)

Basic results in the theory of propagation of elastic-plastic waves have been obtained for the case of one-dimensional motion.

The studies of wave propagation taking into account two displacements are now of greater importance in technical applications and experiments. In particular, analysis of the propagation of elastic-plastic waves due to combined loading is of great interest in the study of the laws of dynamic strength of materials.

At present many papers are being devoted to the experimental and theoretical study of the laws of strength of materials allowing for combined loading with static load application. There are neither experimental nor theoretical studies of combined loading with dynamic load application.

The solution of two problems of propagation of elastic-plastic waves allowing for combined loading will be analysed below.

1. The problem of the compression-shear impact of two free slabs. Assume that two plates of elastic-plastic material collide with each other end-face to end-face. Assume that the velocities of the plates are directed in the planes of and are not normal to the impacting faces (Fig. 1).

In this case there are of course two types of impact possible:

(1) The displacements of the prisms at the impacting faces are equal, or the shear stresses at these faces are smaller than the maximum frictional stress τ_{max};

(2) The shear stresses on the impacting faces are equal to τ_{max}. In this case, the displacements at these faces are different, and there is

a discontinuity.

Owing to the appearance of shear stresses on the impacting faces, compression waves as well as shear waves will arise in the prisms.

The general dynamic equations of a continuous medium have the well-known form:

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}$$

$$\rho\frac{\partial^2 v}{\partial t^2}=\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z} \qquad (1.1)$$

$$\rho\frac{\partial^2 w}{\partial t^2}=\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}$$

In these equations, u, v, and w are displacements, and X_x, ..., Z_z are components of the stress tensor. Assume that $Z_z = 0$ (Fig. 1) everywhere, and $Z_x = Z_y = 0$ only at the boundaries. Assume also that the dimensions of the plates in the direction of the Oy axis are sufficiently large in comparison with the dimensions in the directions Ox, Oz.

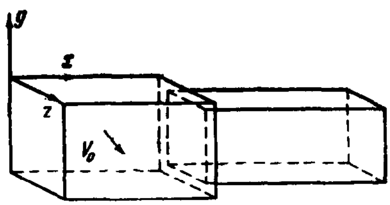

Fig. 1.

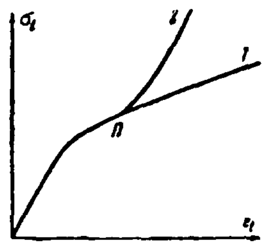

Fig. 2.

Integrating both parts of (1.1) with respect to z from 0 to h, we obtain

$$\rho\frac{\partial^2 u^\circ}{\partial t^2}=\frac{\partial X_x{}^\circ}{\partial x}+\frac{\partial X_y{}^\circ}{\partial y}, \qquad \rho\frac{\partial^2 v^\circ}{\partial t^2}=\frac{\partial Y_x{}^\circ}{\partial x}+\frac{\partial Y_y{}^\circ}{\partial y} \qquad (1.2)$$

In these equations the superscript ^{0} next to a quantity denotes an average along the Oz axis:

$$u^\circ=\frac{1}{h}\int_0^h u(x, z, t)\,dz, \qquad v^\circ=\frac{1}{h}\int_0^h v(x, z, t)\,dz$$

hence

$$u^\circ=u^\circ(x, t), \qquad v^\circ=v^\circ(x, t)$$

We can now write the laws of elastic-plastic deformations for the quantities averaged:

$$X_x^\circ - \sigma^\circ = \frac{2}{3}\frac{\sigma_i^\circ}{e_i^\circ}(e_{xx}^\circ - e^\circ), \qquad X_y^\circ = \frac{1}{3}\frac{\sigma_i^\circ}{e_i^\circ}e_{xy}^\circ$$

$$Y_y^\circ - \sigma^\circ = \frac{2}{3}\frac{\sigma_i^\circ}{e_i^\circ}(e_{yy}^\circ - e^\circ), \qquad Y_z^\circ = \frac{1}{3}\frac{\sigma_i^\circ}{e_i^\circ}e_{yz}^\circ \tag{1.3}$$

$$Z_z - \sigma^\circ = \frac{2}{3}\frac{\sigma_i^\circ}{e_i^\circ}(e_{zz}^\circ - e^\circ), \qquad \sigma = 3ke^\circ$$

These relations will be valid in the region both of uniaxial and of two-dimensional state of stress, but the functions $\sigma_i^\circ = \sigma^\circ(e_i^\circ)$ will be different (Fig. 2).

The position of the transition point (point П, Fig. 2) depends only on the intensity of the compression wave, but the character of the curve in the region of two-dimensional state of stress may depend on the compressive and the shear stresses as well as on the magnitude of the mean hydrostatic pressure σ.

By virtue of the symmetry of the problem we have

$$u^\circ = u^\circ(x, t), \qquad v^\circ = v^\circ(x, t)$$

and therefore, in (1.2)

$$\frac{\partial X_y}{\partial y} = \frac{\partial Y_y}{\partial y} = 0$$

In (1.3) the two equations before the last reduce to identities.

Since u° and v° is independent of y, we get

$$e_{yy}^\circ = 0 \tag{1.4}$$

We also have

$$Z_z = 0$$

Thus in the five equations (1.3), since two of them reduce to identities, the unknowns will be X_x°, Y_y°, X_y°, e_{zz}, i.e. four in number.

Nevertheless there will be no contradiction, since the third equation of (1.3) is a consequence of the first two.

Actually, because $e_{yy}^\circ = 0$ and $Z_z^\circ = 0$, we get

$$e^\circ = \frac{1}{3}(e_{xx}^\circ + e_{zz}^\circ), \qquad \sigma = \frac{1}{3}(X_x^\circ + Y_y)$$

Therefore the first two equations of (1.3) yield

$$\frac{2}{3} X_x{}^\circ - \frac{1}{3} Y_y = \frac{2}{3} \frac{\sigma_i{}^\circ}{e_i{}^\circ} \left(\frac{2}{3} e_{xx}{}^\circ - \frac{1}{3} e_{zz}{}^\circ\right)$$

$$\frac{2}{3} Y_y{}^\circ - \frac{1}{3} X_x{}^\circ = \frac{2}{3} \frac{\sigma_i{}^\circ}{e_i{}^\circ} \left[-\frac{1}{3}\left(e_{xx}{}^\circ - \frac{1}{3} e_{zz}{}^\circ\right)\right]$$

Combining these two equations, we get

$$-\frac{1}{3}(X^\circ + Y_y{}^\circ) = \frac{2}{3} \frac{\sigma_i{}^\circ}{e_i{}^\circ} \left[\frac{1}{3} e_{xx}{}^\circ - \frac{2}{3} e_{zz}{}^\circ\right]$$

On the other hand, the third equation of (1.3) yields

$$-\frac{1}{3}(X^\circ + Y^\circ) = \frac{2}{3} \frac{\sigma_i{}^\circ}{\sigma_i{}^\circ} \left(\frac{2}{3} e_{zz}{}^\circ - \frac{1}{3} e_{xx}{}^\circ\right)$$

As we see, our assertion has been proved. We shall thus'have

$$\frac{2}{3} X_x{}^\circ - \frac{1}{3} Y_y{}^\circ = \frac{2}{3} \frac{\sigma_i{}^\circ}{e_i} \left(\frac{2}{3} e_{xx}{}^\circ - \frac{1}{3} e_{zz}{}^\circ\right)$$

$$-\frac{1}{3}(X_x{}^\circ + Y_y{}^\circ) = \frac{2}{3} \frac{\sigma_i{}^\circ}{e_i} \left(\frac{2}{3} e_{zz}{}^\circ - \frac{1}{3} e_{xx}{}^\circ\right) \qquad (1.5)$$

$$\frac{1}{3}(X_x{}^\circ + Y_y{}^\circ) = k\,(e_{xx}{}^\circ + e_{zz}{}^\circ)$$

$$e_i{}^\circ = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x} - e_{zz}\right)^2 + e_{zz}{}^2 + \frac{2}{3}\left(\frac{\partial v}{\partial x}\right)^2}$$

From these equations one can determine $X_x{}^\circ$ and $Y_y{}^\circ$ as functions of $\partial u^\circ/\partial x$, and $\partial v^\circ/\partial y$ for the region where the stress is assumed to be that of compression as well as shear.

Equations (1.2) will of course have the form

$$\frac{\partial^2 u}{\partial t^2} = a_1{}^2 \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 v}{\partial x^2}, \qquad \frac{\partial^2 v}{\partial t^2} = a_2{}^2 \frac{\partial^2 v}{\partial x^2} + b_2 \frac{\partial^2 u}{\partial x^2} \qquad (1.6)$$

In the uniaxial stress region $v \equiv 0$, and we shall have the following equation of motion:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (1.7)$$

The boundary between the uniaxial stress region and the two-dimensional stress region in the plane of the variables x and t will be a straight line at least up to the time of reflection of the waves from the free faces of the impacting bodies.

Let the equation of this straight line be

$$x = bt \qquad (1.8)$$

This straight line is a wave of strong discontinuity. In front of it there will move systems of continuous Riemann waves, which arise from the solution of equation (1.7). This solution, as can easily be seen, yields the integral

$$u_{1t} = \int_0^{u_x} a_1(u_{1x})\,du = -\Psi(u_{1x}) \tag{1.8'}$$

Behind the wave of strong discontinuity there will be a region with constant parameters. The behavior of the motion can be better described by turning to Fig. 3. In the region of the Riemann waves *I* the parameters of motion change continuously and satisfy equation (1.7). In region *III* the parameters of motion are constant and equal to their values on *Ot*.

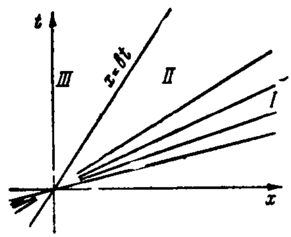

Fig. 3.

The transition from region *III* to region *II* is accomplished by a jump ($x = bt$ is the wave of strong discontinuity), and in the latter region the parameters of motion are also constant. The wave $x = bt$ is the wave of combined loading.

Equation (1.6) should be satisfied in region *III*.

Since in this region the parameters of motion are constant, (1.6) is satisfied identically.

It thus appears unnecessary to know the $\sigma_i = \sigma_i(e_i)$ diagram for the analysis of the motion. It will be shown below that this is not so.

At the wave of strong discontinuity, compatibility equations containing deformation rates and stresses must be fulfilled. When solving these equations it becomes necessary to use the theory of plastic deformations.

At the front of the strong discontinuity the displacements u and v should be continuous, i.e.

$$u_3(x, t) = 0, \qquad u_3(x, t) = u_1(x, t)$$

Differentiating these equations along the wave of strong discontinuity yields

$$v_{3t} + bv_{3x} = 0, \qquad u_{3t} + bu_{3x} = u_{1t} + bu_{1x} \tag{1.9}$$

The momentum equations will give

$$b\rho v_{3t} = -X^{\circ}_{y3}, \qquad b\rho (u_{3t} - u_{1t}) = X_{x3}{}^{\circ} - X_{x1}{}^{\circ} \tag{1.10}$$

For the case of no slipping, at the impacting faces, i.e. at $x = 0$, we shall have

$$u_{3t} = u_0, \qquad v_{3t} = v_0 \tag{1.11}$$

where u_0 and v_0 are the given velocities.

In the presence of slipping, however, on the impacting faces we shall have

$$u_{3t} = u_0, \qquad X_{x3}{}^{\circ} = \tau_{\max} \tag{1.11'}$$

Let us recall that these equations should supplement the Riemann integrals (1.8).

Since the stresses $X^{0}{}_{x3}$, $X^{0}{}_{y3}$, $X^{0}{}_{x}$ and the deformations u_{3x}, v_{3x} should satisfy equations (1.5), the stresses can be expressed in terms of deformations from that equation.

By virtue of this fact, seven equations (1.8), (1.9), (1.10), (1.11), and (1.11') will serve for the determination of the seven quantities: v_{3t}, u_{3t}, u_{1t}, b, v_{3x}, u_{3x}, u_{1x}.

Finally we shall note that when solving system (1.5) the stress intensities σ_{i3} and σ_{i1} appearing there will differ in their dependence on their arguments, since the transition through the discontinuity wave represents a combined loading.

2. The case of elastic deformations. For elastic deformations $\sigma_i = E e_i$, and thus (for the sake of simplifying the notation the superscripts have been dropped) the system (1.5) becomes:

$$\begin{gathered}
\tfrac{2}{3} X_x - \tfrac{1}{3} Y_y = \tfrac{2}{3} E \left[\tfrac{2}{3} \frac{\partial u}{\partial x} - \tfrac{1}{3} e_{zz}\right] \\
-\tfrac{1}{3}(X_x + Y_y) = \tfrac{2}{3} E \left[\tfrac{2}{3} e_z - \tfrac{1}{2} \frac{\partial u}{\partial x}\right] \\
\tfrac{1}{3}(X_x + Y_y) = k\left(\frac{\partial u}{\partial x} + e_{zz}\right)
\end{gathered} \tag{2.1}$$

From this follows

$$X_x = \tfrac{2}{3} E \left(\frac{\partial u}{\partial x} - e_{zz}\right), \qquad -k\left(\frac{\partial u}{\partial x} + e_{zz}\right) = \tfrac{2}{3} E \left(\tfrac{2}{3} e_{zz} - \tfrac{1}{3} \frac{\partial u}{\partial x}\right)$$

or

$$X_x = \frac{2}{3} E (1 - \beta) \frac{\partial u}{\partial x} = \alpha \frac{\partial u}{\partial x}, \qquad \beta = \frac{\frac{1}{3} - \frac{3}{2}\frac{k}{E}}{\frac{2}{3} + \frac{3}{2}\frac{k}{E}} \tag{2.2}$$

The following expression is obtained for the stress X_y:

$$X_y = \frac{1}{3} E \frac{\partial v}{\partial x} = G \frac{\partial v}{\partial x} \tag{2.3}$$

The equations of motion (1.2) will be the following:

$$\rho \frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad \rho \frac{\partial^2 v}{\partial t^2} = G \frac{\partial^2 v}{\partial x^2} \tag{2.4}$$

In the given case instead of (1.8) we obtain

$$u_{1t} = -\sqrt{\frac{\alpha}{\rho}} \tag{2.5}$$

$$v_{3t} + bv_{3x} = 0, \qquad u_{3t} + bu_{3x} = u_{1t} + bu_{1x} \tag{2.6}$$

From the first equation in (1.10), on the basis of (2.3), we get

$$b\rho v_{3t} = -Gv_{3x}$$

If the first equation of (2.6) is considered, we get

$$b^2 = \frac{G}{\rho} \tag{2.7}$$

As is to be expected, the wave of strong discontinuity moves with the velocity of so-called transverse waves.

The second equation of (1.10) yields

$$b\rho (u_{3t} - u) = \alpha (u_{3x} - u_{1x}) \tag{2.8}$$

From (2.6) we get

$$u_{3t} - u_{1t} = -b (u_{3x} - u_{1x}) \tag{2.9}$$

Equations (2.8) and (2.9) are linear homogeneous equations in terms of differences of velocities and deformations. The determinant of this system is not equal to zero, and thus we get

$$u_{3t} = u_{1t}, \qquad u_{3x} = u_{1x}$$

Thus we have shown that the longitudinal velocities and deformations are continuous. By the same token it has been shown that with elastic deformations the presence of shear waves does not change the character of the propagation of longitudinal waves. The law of the independence of the action of forces is valid.

3. The problem of compression-shear impact of two slabs lying between rigid planes. The problem analyzed in Section 1 was solved approximately for the case of plastic deformations, because rigorous study of the averaged values was impossible. An exact solution of the dynamic equations of the theory of plasticity when one of the displacements is limited beforehand, will now be given.

Hence, let $w \equiv 0$ and $u = u(x)$ and $v = v(x)$. Obviously, the condition $w \equiv 0$ will be satisfied if the plastic material lies between two absolutely rigid walls. With the assumed displacements we get $e_{xz} = e_{yz} = 0$, and consequently $Y_z = X_z$.

Moreover, with $e = \partial u / \partial x$ we have

$$X_x = 3k \frac{\partial u}{\partial x}, \qquad Y_y = 3k \frac{\partial u}{\partial x} + \frac{2}{3} \frac{\sigma_i}{e_i} \left(- \frac{\partial u}{\partial x}\right)$$

$$X_y = \frac{1}{3} \frac{\sigma_i}{e_i} \frac{\partial v}{\partial x}, \qquad Z_y = 3k \frac{\partial u}{\partial x} + \frac{2}{3} \frac{\sigma_i}{e_i} \left(- \frac{\partial u}{\partial x}\right) \tag{3.1}$$

$$e_i = \sqrt{2 \left(\frac{\partial u}{\partial x}\right)^2 + \frac{3}{2} \left(\frac{\partial v}{\partial x}\right)^2}$$

As we see, all four stresses X_x, Y_y, Z_z, Z_y are determined as functions of $\partial u / \partial x$ and $\partial v / \partial x$ and are functions of x only.

In this case the equations of motion become

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x}, \qquad \rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial Y_x}{\partial x} \tag{3.2}$$

Substitution from (3.1) yields

$$\rho \frac{\partial^2 u}{\partial t^2} = a_1^2 \frac{\partial^2 u}{\partial x^2} + b_1 \frac{\partial^2 v}{\partial x^2}, \qquad \rho \frac{\partial^2 v}{\partial t^2} = a_2^2 \frac{\partial^2 u}{\partial x^2} + b_2 \frac{\partial^2 v}{\partial x^2} \tag{3.3}$$

These equations are analogous to equations (1.6), and thus the problem of the impact of slabs can be solved quite analogously as in Section 1.

Note that the experimental realization of the conditions of this problem is entirely possible, if a soft metal is used for the impacting slabs and the impact takes place between hardened steel plates.

All problems analysed above are directly analogous to the problem of transverse impact upon a flexible coupling. This permits the complete use of the experimental data mentioned in this paper for the study of transverse impact.

4. Torsion-compression impact. Let us investigate the impact of a hollow cylinder rotating about its axis of symmetry and having a forward velocity along this axis upon another hollow cylinder (Fig. 4).

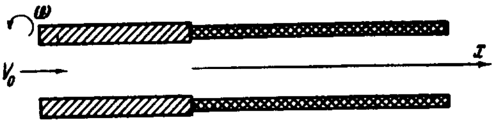

Fig. 4.

Obviously, owing to the presence of friction along the end surface, there will arise on it shear and compressive stresses, which will twist and compress both cylinders. Let us write the equations of motion for a continuous medium in cylindrical coordinates

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial X_x}{\partial x}+\frac{\partial (rX_r)}{\partial r}\frac{1}{r},\qquad \rho\frac{\partial^2 v}{\partial t^2}=\frac{\partial X_\theta}{\partial x}+\frac{1}{r^2}\frac{\partial (r^2R_r)}{\partial r} \tag{4.1}$$

Where u is the axial and v is the circumferential displacement.

Averaging equations (2.1) over the thickness of the cylinder, and assuming that the stresses X_r and R_θ on the surface of the cylinder are equal to zero, we get

$$\rho\frac{\partial^2 u^\circ}{\partial t^2}=\frac{\partial X_x{}^\circ}{\partial x},\qquad \rho\frac{\partial^2 v^\circ}{\partial t^2}=\frac{\partial X_\theta{}^\circ}{\partial x} \tag{4.2}$$

since

$$\int\limits_{r_0-h}^{r_0+h}\frac{1}{2}\frac{\partial (X_r r)}{\partial r}\partial r\approx\frac{1}{r_{\text{cp}}}\int\partial (X_r, r)=0$$

$$\int\limits_{r_0-h}^{r_0+h}\frac{1}{r^2}\frac{\partial (R_\theta r^2)}{\partial r}\partial r\approx\frac{1}{r_{\text{cp}}{}^2}\int\limits_{r_0-h}^{r_0+h}\partial (r^2R_\theta)=0$$

Because $e_{\theta\theta}=0$ and $R_r=0$, we obtain

$$e^\circ=\tfrac{1}{3}(e_{xx}{}^\circ+e_{rr}{}^\circ),\qquad \sigma=\tfrac{1}{3}(X_x+\theta_\theta) \tag{4.3}$$

Analogously to the previous paragraph we get

$$X_x-\sigma^\circ=\frac{2}{3}\frac{\sigma_i{}^\circ}{e_i{}^\circ}(e_{xx}{}^\circ-e^\circ),\qquad \sigma=3ke^\circ.$$

$$-\sigma^\circ=\frac{2}{3}\frac{\sigma_i{}^\circ}{e_r{}^\circ}(e_{rr}{}^\circ-e^\circ),\qquad X_\theta{}^\circ=\frac{1}{3}\frac{\sigma_i{}^\circ}{e_i{}^\circ}e_{x\theta}{}^\circ \tag{4.4}$$

$$e_{xx}{}^\circ=\frac{\partial u^\circ}{\partial x},\qquad e_{x\theta}{}^\circ=\frac{\partial v^\circ}{\partial x} \tag{4.5}$$

Inasmuch as $e_{\theta\theta}=e_{\theta r}=e_{xr}=0$, the deformation intensity has the form

$$e_i = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x} - e_{rr}\right)^2 + e_{rr}^2 + \frac{2}{3}\left(\frac{\partial v}{\partial x}\right)^2} \tag{4.6}$$

From the four equations (4.4) one can determine X_x°, θ_θ°, X_θ°, e_{rr} by using $e_{xx}^\circ = \partial u^\circ/\partial x$ and $e_{x\theta}^\circ = \partial v^\circ/\partial x$.

After this development, equation (4.2) will be of the same form as equation (1.6). Therefore, from the mathematical point of view, the problem has been reduced to the same one as in the compression-shear impact.

Translated by M.I.Y.